Unit-1

n^{th} Differential Coefficient of Various Function

1. If $y = x^m$ then

$$y_n = m(m-1)(m-2)\ldots(m-n+1)\,x^{m-n},$$

where $n < m$

2. If $y = (ax+b)^m, n < m$ then

$$y_n = m(m-1)(m-2)\ldots \ldots(m-n+1) \times (ax+b)^{m-n}\,.\,a^n$$

3. If $y = \dfrac{1}{ax+b}\left[x \neq -\dfrac{b}{a}\right]$ then

8. If $y = e^{ax} \sin(ax + b)$ then

$$y_n = (a^2 + b^2)^{n/2} e^{ax} . \sin\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$$

9. If $y = e^{ax} \cos(ax + b)$ then

$$y_n = (a^2 + b^2)^{n/2} e^{ax} . \cos\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$$

constant, is called partial derivative of z w.r.t. x and is denoted by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x$$

Similarly the derivative of z. with respect to *y*, treating *x* as constant is called partial derivative of z w.r.t. *y* and is denoted by

$$\frac{\partial z}{\partial y} \text{ or } \frac{\partial f}{\partial y} \text{ or } f_y$$

Higher Order Derivatives

IInd Order

$$\frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx},$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Composite Function

There are two types of composite function :

1. If $u = f(x, y)$ when $x = \phi(t)$, $y = \Psi(t)$, then u is called a composite function of (the single variable) t and we can find $\frac{du}{dt}$ by formula

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt}$$

2. If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \Psi(u, v)$ then z is

$$\frac{d^2y}{dx^2} = -\left(\frac{dx}{dy}\right)^{-3} \frac{d^2x}{dy^2},$$

$$\frac{d^3y}{dx^3} = -\left(\frac{dx}{dy}\right)^{-4} \frac{d^3x}{dy^3} + 3\left(\frac{dx}{dy}\right)^{-5}\left(\frac{d^2x}{dy^2}\right)^2$$

2. Change of independent variable x into another variable t where $x = f(t)$, then :

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt}\left(\frac{dx}{dy}\right)^{-1}$$

$$= \left(\frac{dx}{dt}\right)^{-1} \frac{dy}{dt}$$

3. Let $u = f(x, y)$, where

$x = F_1(t_1, t_2)$,

$y = F_2(t_1, t_2)$

then $$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial t_1}$$

and

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial t_2}$$

Maclaurin's Theorem :

Putting $x = 0$ and $h = x$ in Taylor's theorem :

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^n(0)$$

Taylor's Theorem for a Function of Two Variables :

$$f(x+h, y+k) = f(x, y) + (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx}$$

Properties of Jacobians :

1. If u, v are functions of r, s where r, s are functions of x, y, then :

$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{\partial (u, v)}{\partial (r, s)} \times \frac{\partial (r, s)}{\partial (x, y)}$$

2. If J_1 is the Jacobian of u, v with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v then

$$J_1 J_2 = 1 = \frac{\partial (u, v)}{\partial (x, y)} \times \frac{\partial (x, y)}{\partial (u, v)}$$

Extreme Values of a Function (Maxima and Minima) :

Rule 1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

2. Solve $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ simultaneously.

3. For each solution in step (ii) find $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \, \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$.

4. (a) If $rt - s^2 > 0$ and $r < 0$ for particular solution (a, b) of

For $f(x, y, z)$ to have a max. or min. value, the necessary condition is

$$\frac{\partial f}{\partial x} = 0,$$

$$\frac{\partial f}{\partial y} = 0,$$

$$\frac{\partial f}{\partial z} = 0$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$$

and $$A^{-1} = \frac{adj\ A}{|A|}$$

Elementary Transformations

Any one of the following operation on a matrix is called an elementary transformation.

1. Inter-change of any two rows (columns). This transformation is indicated by R_{ij} (C_{ij}), if ith row (column) and jth row (column) are interchanged.
2. Multiplication of the elements of any row R_i (or column C_i) by a

Symbolically rank of $A = r$ is written as $\rho(A) = r$.

Method of Finding Rank

Normal Form Method (Canonical Form)

If A is an $m \times n$ matrix and by a series of elementary (row or column or both operations. It can be put into one of the following forms :

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_r \\ 0 \end{bmatrix}, [I_r \quad 0], [I_r]$$

where I_r is the unit matrix of order r.

is called normal form or canonical form.

diagonal elements when it has been reduced.

Solution of System of Linear Equations

System of non-homogeneous linear equations can be written in matrix form $AX = B$ and follow following rules :

(i) Find rank of augmented matrix $[A : B]$

(a) If $\rho\,[A : B] \neq \rho\,(A)$, the system is inconsistent (having no solution).

(b) If $\rho\,[A : B] = \rho\,(A) =$ no. of variables (unknowns), system is consistent and have a unique solutions.

Corresponding to each root, the homogeneous system has a non-zero solution which is called Eigen vector or latent vector.

Cayley Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

Reduction of a Matrix to Diagonal Form

If a square matrix A of order n has n linearly independent Eigen vectors, then a matrix B can be found such that $B^{-1}AB$ is a diagonal matrix.

Then $\iint_R f(x, y)\, dx\, dy$

$$= \iint_{R_1} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

To change Cartesian coordinates (x, y, z) to cylindrical coordinates (r, θ, ϕ)

We have

$x = r\cos\phi,\ y = r\sin\phi,\ z = z,$

$$J = \frac{\partial(x, y, z)}{\partial(r, \phi, z)} = r$$

$$\iiint_V f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_V f(r\cos\phi, r\sin\phi, z)\, r\, dr\, d\phi\, dz$$

(iv) For Pedal Equation :

Length of arc

$$= \int_a^b \frac{r}{\sqrt{r^2 - p^2}}\, dr$$

Gamma Function :

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx, \quad n > 0$$

Note : 1. $\Gamma(n+1) = n\,\Gamma n$

2. $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

3. $\Gamma 1 = 1$

Beta Function :

If m, n are positive then

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$m > 0, n > 0$$

LIOUVILLE'S EXTENSION OF DIRICHLET THEOREM

If the variables x, y, z are all positive such that

$$h_1 < (x + y + z) < h_2$$

then

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du$$

3. Divergence of a Vector Function :

Let, $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, then

$$\text{div } \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\vec{V}$$

$$\text{div } \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Since $\hat{i}\,.\,\hat{i} = \hat{k}\,.\,\hat{k} = \hat{j}\,.\,\hat{j} = 1$

$\hat{i}\,.\,\hat{j} = \hat{j}\,.\,\hat{k} = \hat{k}\,.\,\hat{i} = 0$

4. Curl of a Vector Point Function :

Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, then

$$\text{Curl } \vec{V} = \nabla \times \vec{V}$$

Stoke's Theorem :

(Relation between line and surface integral).

If S is an open surface bounded by a closed curve C and $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is any vector point function having continuous first order partial derivatives, then :

$$\oint_C \vec{F}.d\vec{r} = \iint_S curl\ \vec{F}\ .\ \hat{n}\ dS$$

where $\hat{n}$ is the unit normal vector drawn at any point S in the sense in which a right handed screw would advance when rotated in the sense of description of C.